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RESEARCH ARTICLE

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Beyond the linearity critique: the knife-edge assumption of steady-state growth

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Abstract The "linearity critique" of endogenous growth models is presented in a general context of an arbitrary growth model and reassessed. It is argued that presence of linearities is not a valid criterion for rejecting growth models. Existence of exponential/geometrical steady-state growth (i.e. of a balanced growth path with strictly positive growth rates) necessarily requires some knife-edge condition which is not satisfied by typical parameter values. Hence, balanced growth paths are fragile and sensitive to smallest disturbances in parameter values. Adding higher order differential/difference equations to a model does not change the knife-edge character of steady-state growth.

Keywords Long-run economic growth · Knife-edge condition · Balanced growth path · Linearity critique

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1 Introduction

The starting point of this paper is the "linearity critique", in its appealing form due to Jones (2005). Jones argues that all growth models ever discussed in literature

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which were able to deliver steady-state growth, necessarily contained at least one linear differential/difference equation. This is criticized, because strict linearity is a knife-edge assumption, equivalent to requesting that in an equation of form

$$\dot{X} = \alpha X^{\phi}, \tag{1}$$

the parameter ϕ value *exactly* equals unity. Apparently, the long-run dynamics of the X variable would be qualitatively different if $\phi < 1$ or $\phi > 1$.

For the sake of clarity, let us now and until the end of this paper, define a "knife-edge condition" as a condition imposed on parameter values, such that the set of values satisfying this condition has an empty interior in the space of all possible values. Parameter values that are requested to satisfy a particular knife-edge condition would also be referred to in text as "non-typical".

An important remark is that we will find knife-edge conditions in the form of equality constraints. Hence, if the parameter space is finite dimensional (a subset of $\mathbb{R}^n$), the set of values satisfying a knife-edge condition would automatically be of Lebesgue measure zero in the given space.

The main point of this paper is to inspect the following claim due to Jones (2005, p. 62):

> What is not sufficiently well-appreciated, however, is that *any* model of sustained exponential growth requires such a knife-edge condition. Neoclassical growth models [as opposed to endogenous growth models – *J.G.*] are not immune to this criticism; they just assume the linearity to be completely unmotivated.

This claim shall be supported by an explicit proof, that it is the steady-state growth property itself, that is "knife-edge" or requires "non-typical" parameter values.[1] We shall generalize the standard growth framework and prove that even introducing differential/difference equations of an arbitrary finite order does not solve the fundamental problem: sustained exponential/geometrical growth (i.e., existence of a balanced growth path with strictly positive growth rates) *necessarily* relies upon knife-edge assumptions.

Our finding supports the third of Temple's (2003) five "obvious" rules for thinking about long-run growth: "Do not dismiss a model of growth because the long-run outcomes depend on knife-edge assumptions". Indeed, we claim that there is no alternative if one wants the model to possess the steady-state growth property. Thus, this paper adds a further qualification to Temple's view that the issue of knife-edge assumptions in growth models does not deserve the attention it has recently received.[2] No discrimination between models can be made on these grounds.

However, given that all growth models share knife-edge assumptions, and therefore, smallest parameter shifts are enough to reverse their long-run dynamics and eliminate steady-state growth—we come to a pessimistic conclusion that growth is very fragile.

To reinforce our argumentation, let us also point out that the knife-edge character of the "$\phi = 1$ in equation (1)"-type assumptions consists not only in the

[1] Exponential growth is strikingly consistent with some of the well-documented empirical data, however. See, e.g., the figure on page 42 in Jones (2005).

[2] Knife-edge conditions in economic growth models have been reviewed and classified by, among others, Eicher and Turnovsky (1999), Jones (1999), Li (2000), and Christiaans (2004).

fact that the set of parameter values satisfying them has an empty interior (or is of Lebesgue measure zero) in the set of all possible parameter values, but also in the fact that they bound away from each other cases of qualitatively different dynamic behavior of the model. Most likely, these would be explosive cases ($\phi > 1$), and cases where the growth rates gradually fall down to zero ($\phi < 1$).

To our best knowledge, the fundamental result of this paper has not yet been proven in its generality. Closely related literature includes Christiaans (2004) who provides the proof for the specific case of continuous-time models with three state variables (physical capital, technology, population) and differential equations of first order only; and Laffargue (2004) who proves it for discrete-time models with up to two lags, while concentrating on a completely different issue—solving macroeconometric models with perfect foresight—and thus neither attaches any particular weight to the mathematical result, nor gives an interpretation for it.

In the following section, we rephrase the "linearity critique" in the context of a generalized growth model. In section 3, we prove the general theorem that steady-state growth is a knife-edge assumption. Section 4 contains further discussion and concluding remarks.

2 The "singularity" critique

Let us consider a generalized continuous-time model of economic growth. The variant we are going to analyze is standard in the sense that its steady-state properties are determined by a system of *first order* autonomous differential equations of the form

$$\hat{X} = F(X), \quad X(0) \text{ given.} \tag{2}$$

By $X = (X_1, X_2, \ldots, X_n)$ we denote a vector of n state variables. Each ith variable X_i is assumed to be twice continuously differentiable with respect to time. By $\dot{X}$ we denote a vector of X_i's first order time derivatives, and by $\hat{X}$ we denote a vector of their first order log-time derivatives (growth rates). It is assumed that all X_i's are strictly positive.[3] We shall concentrate on autonomous differential equations only, since it is natural for economists to look for general laws that are valid irrespective of time.

A further remark is that in (2), we ignore control (choice, decision) variables. Although these are vital ingredients of economic models which include optimization—as most contemporary growth models do—they can be ruled out from present considerations, since we are interested in the long-run dynamics only.

Assuming that $F \in C^1(\mathbb{R}^n)$ and differentiating both sides of (2) with respect to time yields

$$\dot{\hat{X}} = DF(X) \cdot \dot{X}. \tag{3}$$

The right hand side of (3) is linear with respect to $\dot{X}$. We shall adopt the following definition of a steady state: it is a state in which all growth rates $\hat{X}_i$ are constant.

[3] One can argue that taking (2) already limits the analysis, since we assume that the general equation $\Phi(X, \dot{X}) = 0$ can be solved explicitly for $\dot{X}$. This problem can be resolved for almost all $\dot{X}$ using the Implicit Function Theorem, however. Since this procedure is not revealing and sometimes cumbersome, we purposefully limit ourselves to (2).

Imposing a steady state is equivalent to setting all $\dot{\hat{X}}_i$'s equal to zero. Observing that in the steady state, $DF(X) \cdot \dot{X} = 0$, we state the "linearity critique":

$$\dot{X} = 0 \quad \text{or} \quad DF(X) \text{ is singular.} \tag{4}$$

From (4), we see that the "linearity" critique discussed in literature should rather be called "singularity" critique: it is *singularity* of the $DF(X)$ matrix that is inevitable if one wants to obtain positive steady-state growth in any state variable. Namely, for models propelled by first order autonomous differential equations, exponential growth requires fulfilling the following knife-edge condition: in the steady state, the determinant of the $DF(X)$ matrix (i.e., the Jacobian of the F mapping) *exactly* vanishes.[4]

Note that here, as opposed to, e.g., Jones (1999) who only considers Cobb-Douglas functions with a finite number of parameters (say p), so that the knife-edge conditions are imposed on a subset of $\mathbb{R}^p$, our parameter space is $C^1(\mathbb{R}^n)$—which is an infinite dimensional function space. Lebesgue measure cannot be applied on such space, but it remains clear that the set $\mathcal{F} \subset C^1(\mathbb{R}^n)$ of functions which have a zero Jacobian everywhere along the time path of X, has an empty interior. To show this, let $F \in \mathcal{F}$. For every $\varepsilon > 0$ there exists a function $F_\varepsilon \in C^1(\mathbb{R}^n)$ such that $||F_\varepsilon - F||_{C^1(\mathbb{R}^n)} < \varepsilon$ and $\det(DF_\varepsilon(X)) \neq 0$ for some X along its time path $\{X(t)\}_{t=0}^{\infty}$.[5] Thus, $F_\varepsilon \notin \mathcal{F}$, so $\mathcal{F}$ has an empty interior.

Let us now manipulate (3) to obtain a direct formula for $\hat{X}$. We shall multiply all terms in each ith column ($i = 1, 2, \ldots, n$) of the $DF(X)$ matrix by $X_i > 0$ and denote the resultant matrix by $\Psi(X)$. Notice that

$$\det \Psi(X) = \left(\prod_{i=1}^{n} X_i \right) \det DF(X). \tag{5}$$

It follows that $\det \Psi(X) = 0 \Leftrightarrow \det DF(X) = 0$, i.e., the singularity/non-singularity of $DF(X)$ is inherited by $\Psi(X)$. It is also obtained that

$$DF(X) \cdot \dot{X} = 0 \quad \Leftrightarrow \quad \Psi(X) \cdot \hat{X} = 0. \tag{6}$$

The $\Psi(X)$ matrix proves useful in the formulation of the following claim.

The claim goes as follows. If the matrix $\Psi(X)$ is singular, there exists a *continuum* of $\hat{X}$ vectors—steady-state growth rates—located along the eigenspace associated with the zero eigenvalue of $\Psi(X)$.[6]

[4] A slightly less general result than this can be found in Christiaans (2004), Proposition 1.

[5] One may wonder how to construct the F_ε function. For example, one could apply the following procedure. First, fix an X from the time path such that $||X|| > 1$. This is always possible, because we have assumed exponential growth in at least one state variable. Denote $M = 2||X||$, and then take $F_\varepsilon(X) = F(X) + (\varepsilon/M)X$. Observe that $DF_\varepsilon(X) = DF(X) + (\varepsilon/M)I$. In result, $||F_\varepsilon - F||_{C^1(\mathbb{R}^n)} = ||F_\varepsilon - F||_\infty + ||D(F_\varepsilon - F)||_\infty = (\varepsilon/M)||X|| + (\varepsilon/M)||I||_\infty = (\varepsilon/2) + (\varepsilon/M) < \varepsilon$. Moreover, $\det(DF_\varepsilon(X)) = \det(DF(X) + (\varepsilon/M)I) \neq 0$ unless $-(\varepsilon/M)$ is an eigenvalue of $DF(X)$. Limiting the scope of our reasoning to ε's small enough (in the end, we are interested only in arbitrarily small ε's anyway) rules out this unwanted possibility and thus guarantees $\det(DF_\varepsilon(X)) \neq 0$.

[6] As an example, take a family of Jones' (1995) models, indexed by the population growth rate $n \geq 0$.

If parameters of the model happen to fulfill multiple knife-edge conditions, so that the zero eigenvalue is multifold, then this eigenspace is of a dimension higher than 1.[7]

Let us again emphasize that the parameter values, satisfying the singularity condition are non-typical, that is, the set of such values has an empty interior in the space of all possible values. In consequence, we may say that if there is randomness in their determination, then the singularity condition is typically violated.

What is more, if the F function is not Cobb–Douglas, then $\Psi(X)$ depends on X. Thus, as the variables contained in X grow exponentially over time, the singularity condition still needs to be satisfied for *all* X along the balanced growth path. This accounts for an indeed stringent condition on parameter values and at the same time disqualifies many functional forms of F.

This corollary actually builds up to a version of Uzawa's (1961) Steady-State Growth Theorem: if the production function is not Cobb-Douglas, then while the state variables (X_i's) grow at a constant rate, partial derivatives in DF's columns should change proportionately, in a way that the determinant of $DF(X)$ remains zero at all times.

3 The knife-edge assumption of steady-state growth

The results of the previous section can be generalized. We shall consider models both in continuous and in discrete time. We shall also include differential/difference equations of an arbitrary finite order. Although this point may seem overly theoretical—economic growth models typically include only first-order equations[8]—it remains worth emphasizing that there does not exist a way to circumvent knife-edge assumptions and yet obtain positive long-run growth rates.

Intuitively, the above point is true because of the distinguishing feature of an exponential function, or equivalently, a geometrical sequence. An exponential function is a linear function of its derivative; a geometrical sequence, multiplied by a constant, remains a geometrical sequence.

3.1 The theorem (continuous-time version)

We shall now prove the following general theorem.

Theorem 1 *It is impossible to construct a model, propelled by differential equations, in which there exists a steady state that*

a) implies that some state variables grow exponentially at a constant positive rate,
b) is obtained without knife-edge (singularity) assumptions.

Proof We shall assume the former and show impossibility of the latter.

[7] As an example, take a family of neoclassical growth models, indexed by the population growth rate $n \geq 0$ and the exogenous technology growth rate $g \geq 0$.

[8] It is straightforward to borrow an interpretation of second time derivatives from classical physics: we would not only be talking about the *pace* of accumulation of a certain stock variable, but also about its *acceleration*.

For the sake of the proof, let us consider a generalized type (2) differential model that includes higher order time derivatives of the state variables, contained in the vector $X = (X_1, X_2, \dots, X_n)$. Each ith variable X_i is assumed to be at least $m + 1$ times continuously differentiable with respect to time. By $X^{(k)}$, we shall denote the vector of kth time derivatives of X. The model reads:

$$X^{(m)} = \Phi(X, \dot{X}, \dots, X^{(m-1)}), \quad X(0), \dot{X}(0), \dots, X^{(m-1)}(0) \text{ given}, \tag{7}$$

where m is an arbitrary positive integer.[9] The mapping $\Phi : \mathbb{R}^{mn} \to \mathbb{R}^n$ is assumed to be at least once continuously differentiable. We shall consider the whole $C^1(\mathbb{R}^{mn}; \mathbb{R}^n)$ function space to be our "parameter space".

According to a theorem fundamental to differential equations (see Arnold 1975), we can decompose our system (7) into a system of mn differential equations, $(m-1)n$ of which linear, and write

$$\begin{aligned} \dot{X} &= Y_1 \\ \dot{Y}_1 &= Y_2 \\ &\vdots \\ \dot{Y}_{m-2} &= Y_{m-1} \\ \dot{Y}_{m-1} &= \Phi(X, Y_1, Y_2, \dots, Y_{m-1}). \end{aligned} \tag{8}$$

Now, we shall proceed with the derivations for each ith state variable separately ($i = 1, 2, \dots, n$). We observe that the exponential growth condition requires

$$\dot{\hat{X}}_i = 0 \quad \Rightarrow \quad \dot{Y}_{1,i} X_i = \dot{X}_i Y_{1,i} \quad \Rightarrow \quad \hat{Y}_{1,i} = \hat{X}_i. \tag{9}$$

If $\hat{X}_i = 0$, then slightly abusing notation, we write $\hat{Y}_{k,i} = 0$ for all k as well. By forward recursion, it is automatically obtained that $\hat{X}_i = \hat{Y}_{1,i} = \cdots = \hat{Y}_{m-1,i}$. We shall now pass to the last equation.

Imposing that $\dot{\hat{Y}}_{m-1,i} = 0$ yields

$$\begin{aligned} \hat{X}_i \Phi_i &= \hat{Y}_{m-1,i} \Phi_i \\ &= \frac{\mathrm{d}\Phi_i}{\mathrm{d}t} = D_X \Phi_i \cdot \dot{X} + D_{Y_1} \Phi_i \cdot \dot{Y}_1 + \cdots + D_{Y_{m-1}} \Phi_i \cdot \dot{Y}_{m-1}, \end{aligned} \tag{10}$$

where Φ_i denotes the ith coordinate function of the Φ mapping, $D_X \Phi_i$ denotes the vector of n first order partial derivatives of Φ_i with respect to all X_i's. The same notational convention applies to $Y_{k,i}$'s. Arguments of the functions have been omitted for convenience.

With all vectors $D_X \Phi_i, D_{Y_1} \Phi_i, \dots, D_{Y_{m-1}} \Phi_i$, we redo the same algebraic manipulations as we did with DF in (3), that is, we multiply each jth element of the vector by X_j, and $Y_{k,j}$ respectively, and denote the resulting vector by $\Psi_{k,i}$, where $k = 0, 1, 2, \dots, m-1$. It is then obtained that

$$\hat{X}_i \Phi_i = \Psi_{0,i} \cdot \hat{X} + \Psi_{1,i} \cdot \hat{Y}_1 + \cdots + \Psi_{m-1,i} \cdot \hat{Y}_{m-1} \tag{11}$$

$$\Downarrow \quad (\text{as } \hat{X} = \hat{Y}_1 = \cdots = \hat{Y}_{m-1})$$

$$\Psi_{\Sigma,i} \cdot \hat{X} = 0, \tag{12}$$

[9] Again, it is possible to get an explicit solution for $X^{(m)}$ locally almost everywhere. The Implicit Function Theorem is used to prove this claim.

where the vector $\Psi_{\Sigma,i} = \Psi_{0,i} + \cdots + \Psi_{m-1,i} - \Phi_i \mathbf{e}_i$, and $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the ith unit vector.

We now combine these n symmetrical results to get the final result:

$$\Psi_\Sigma \cdot \hat{X} = 0, \tag{13}$$

where we have denoted

$$\Psi_\Sigma = \begin{pmatrix} \Psi_{\Sigma,1} \\ \Psi_{\Sigma,2} \\ \vdots \\ \Psi_{\Sigma,n} \end{pmatrix} = \begin{pmatrix} \Psi_{0,1} + \cdots + \Psi_{m-1,1} - \Phi_1 \mathbf{e}_1 \\ \Psi_{0,2} + \cdots + \Psi_{m-1,2} - \Phi_2 \mathbf{e}_2 \\ \vdots \\ \Psi_{0,n} + \cdots + \Psi_{m-1,n} - \Phi_n \mathbf{e}_n \end{pmatrix}. \tag{14}$$

Since we assumed $\hat{X} \neq 0$, the $n \times n$ matrix Ψ_Σ is necessarily singular.

To prove that singularity is indeed a knife-edge condition, we take $\mathcal{F} \subset C^1(\mathbb{R}^{mn}; \mathbb{R}^n)$ to be the set of functions Φ such that their associated $\det(\Psi_\Sigma) = 0$. Let $\Phi \in \mathcal{F}$. For every $\varepsilon > 0$ there exists a function $\Phi_\varepsilon \in C^1(\mathbb{R}^{mn}; \mathbb{R}^n)$, together with an associated matrix $\Psi_{\varepsilon,\Sigma}$ such that $||\Phi_\varepsilon - \Phi||_{C^1(\mathbb{R}^{mn};\mathbb{R}^n)} < \varepsilon$ and $\det(\Psi_{\varepsilon,\Sigma}) \neq 0$ for some X along the time path.[10] Thus, $\Phi_\varepsilon \notin \mathcal{F}$, so $\mathcal{F}$ has an empty interior. □

Equation (13) is the central result of this paper, albeit the final "impossibility" conclusion in its core is very much alike (4). If the knife-edge condition that Ψ_Σ be singular is satisfied, there emerge a continuum of steady states along the space spanned by the eigenvectors associated with Ψ_Σ's zero eigenvalue. If the knife-edge singularity condition is *not* satisfied, which happens in the typical case, then the only steady state is the one with zero growth.

Let us also emphasize, that for non-Cobb-Douglas Φ functions, Ψ_Σ becomes a non-trivial function of the vector $(X, \dot{X}, \ldots, X^{(m-1)})$. In such case, the facts that the singularity condition imposed on Ψ_Σ is assumed to hold at *all* times, and that arguments of the Ψ_Σ function evolve in time, together constitute a very stringent condition on Φ's parameters and effectively rule out numerous functional forms of Φ.

As m was arbitrary, the main result applies to systems of differential equations of any finite order.

3.2 The theorem (discrete-time version)

We shall now prove that a version of the above theorem holds also if the model is set up in discrete time.

Theorem 2 *It is impossible to construct a model, propelled by difference equations, in which there exists a steady state that*

a) implies that some state variables grow geometrically at a constant positive rate,
b) is obtained without knife-edge assumptions.

[10] One could again use the family of functions $\Phi_\varepsilon = \Phi + (\varepsilon/M)X$, as in footnote 5.

Proof We shall again assume the former and show impossibility of the latter.

Let us now consider a general discrete-time growth model. Values of the state variables at time $t = -m+1, -m+2, \dots, -1, 0, 1, 2, \dots$ are contained in the vector $X_t = (X_{1,t}, X_{2,t}, \dots, X_{n,t})$. The model can be written as:

$$X_t = \Phi(X_{t-1}, X_{t-2}, \dots, X_{t-m}), \quad X_{-m+1}, X_{-m+2}, \dots, X_0 \text{ given}, \tag{15}$$

where m is some arbitrary positive integer. We do not have to impose any particular assumptions on the Φ mapping. Thus, the space of all mappings $\Phi : \mathbb{R}_+^{mn} \to \mathbb{R}_+^n$ is going to be considered our "parameter space", and denoted by $\mathcal{P}$. We shall endow the space $\mathcal{P}$ with the usual supremum metric but without ruling out functions that are divergent with respect to this metric.

The model (15) is a discrete-time version of the continuous-time model (7).

The assumption of geometrical steady-state growth implies that for each ith state variable X_i, there exists a constant $\gamma_i \in (0, 1]$ such that $X_{i,t-k} = \gamma_i^k \cdot X_{i,t}$ for all $k = 1, 2, \dots$, and for at least one i, $\gamma_i < 1$. If we denote

$$\gamma = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_n \end{pmatrix}, \tag{16}$$

then we can write all n symmetrical equations simultaneously in the compact form $X_{t-k} = \gamma^k \cdot X_t$.

Write (15) for two consecutive time periods t and $t+1$ to see that for each ith state variable, it is also true that

$$\Phi_i(\gamma X_t, \gamma^2 X_t, \dots, \gamma^m X_t) = \gamma_i \Phi_i(X_t, \gamma X_t, \dots, \gamma^{m-1} X_t), \tag{17}$$

where Φ_i denotes the ith coordinate function of the Φ mapping.

Now collect all n equations (17), denote $\Psi(X_t) = \Phi(X_t, \gamma X_t, \dots, \gamma^{m-1} X_t)$ and make use of (16) to see that

$$\Psi(\gamma X_t) = \gamma \cdot \Psi(X_t). \tag{18}$$

Since we assumed $\gamma \neq I$ and $\gamma \neq 0$, the Ψ mapping is necessarily homogenous of degree *exactly* one with regard to the given diagonal γ matrix.

Degree-one homogeneity is a knife-edge condition. To prove it, let $\mathcal{F} \subset \mathcal{P}$ be the set of functions Φ such that their associated Ψ functions are homogenous of degree one for all X_t along the time path $t = 1, 2, \dots$. Take $\Phi \in \mathcal{F}$. For every $\varepsilon > 0$ there exists a function $\Phi_\varepsilon \in \mathcal{P}$, together with its associated Ψ_ε and γ_ε, such that $||\Phi_\varepsilon - \Phi||_{L^\infty(\mathbb{R}_+^{mn};\mathbb{R}_+^n)} < \varepsilon$ and $\Psi_\varepsilon(\gamma_\varepsilon X_t) \neq \gamma_\varepsilon \cdot \Psi_\varepsilon(X_t)$ for some X_t along the time path $\{X_t\}_{t=0,1,2,\dots}$[11] Thus, $\Phi_\varepsilon \notin \mathcal{F}$, so $\mathcal{F}$ has an empty interior. □

Equation (18) is assumed to hold for *all* X_t's along the time path. For non-Cobb-Douglas Ψ functions, it becomes an indeed stringent condition on Φ's parameters and rules out numerous functional forms of Φ.

[11] Again, we refer back to footnote 5 for the idea how to construct such functions Φ_ε. For example, one could use $\Phi_\varepsilon = \Phi + (\varepsilon/M)X_t$.

As m was arbitrary, the result applies to systems of difference equations of any finite order.

At this point, we shall give more intuition on how knife-edge conditions in the form of singularity and degree-one homogeneity correspond to each other. Let us assume that the coordinate functions of Ψ, denoted $\Psi_i(X_t) \equiv \Phi_i(X_t, \gamma X_t, \ldots, \gamma^{m-1} X_t)$ are homogenous of degree one and twice continuously differentiable for all i. Then, their partial derivatives $(\partial \Psi_i / \partial X_{j,t})(X_t)$, $j = 1, 2, \ldots, n$ are homogenous of degree zero and by Euler's theorem on homogenous functions,

$$\sum_{k=1}^{n} \frac{\partial^2 \Psi_i(X_t)}{\partial X_{j,t} \partial X_{k,t}} X_{k,t} = 0. \tag{19}$$

When put in matrix notation, (19) reads $D^2 \Psi_i(X_t) \cdot X_t = 0$. Since $X_t \neq 0$ by assumption, this implies singularity of the Hessian of Ψ_i. Our reasoning is valid for all i, of course.

Hence, we see that in the specific case of homogenous and twice continuously differentiable mappings Φ, *degree-one homogeneity implies singularity* of Hessians of the coordinate functions of Ψ. Please note, however, that the main theorem of this subsection applies to a more general class of mappings Φ than discussed in the above paragraph.

4 Discussion

We shall now briefly discuss some of the theoretical issues arising from the above "impossibility" theorem.[12] We shall identify the consequences of the theorem by confronting it with a few informal rules which seem to be commonly obeyed in contemporary economic growth theory. The list goes as follows:

- growth models are designed to explain (some) historical data on productivity and GDP;
- they should generate steady-state growth in productivity and GDP;
- exogenous technology growth (e.g., in the form of a $\dot{A} = gA$ equation) should be avoided;
- strong scale effects should be eliminated;
- models are appreciated more, if they are able to deliver optimistic predictions. Long-run growth that does not require continued population growth, and does not hinge upon some exogenously growing factor in production, is certainly a desirable outcome.

This list is admittedly simplified, but still able to capture some of the vital characteristics of the paradigm in contemporary growth theory. We shall now juxtapose these points with the fact that steady-state growth is a knife-edge assumption.

Consistency with empirical data is an obvious property of a good economic theory, and an approximately exponential growth in productivity and GDP in twentieth-century U.S.A. is part of the evidence. The most popular (and natural) way

[12] I would like to thank Charles Jones for shifting my attention to some of these discussion points.

to obtain long-run exponential growth of such variables in a formal model is to make it a steady-state property. "Long run" is then readily identified with the steady state, and "short run" – with the transitional dynamics. In such models, however, knife-edge assumptions are necessary.

The desire to avoid exogenous technology growth gave birth to a wide variety of models, in which growth is driven endogenously, i.e., via purposeful human capital accumulation, R&D expenditure, etc. The original linearity assumption has been shifted from the $\dot{A} = gA$ equation to other equations of the model, or disguised among multiple variables.

Building endogenous growth models without strong scale effects leads either to semi-endogenous theories in which the long-run growth rate is pinned down by the exogenous population growth rate (e.g., Jones 1995), or to endogenous growth models "of the second generation" (e.g., Young 1998) which require multiple knife-edge conditions.

The last point of the above list is affected by the theorem in the most obvious way. If one refrains from making knife-edge assumptions, she will no longer be able to obtain the optimistic prediction of sustained growth.

We shall also address the question whether taking demographics as exogenously given (external to the economy) can alleviate the problem of knife-edge assumptions. In the light of the main theorem of this paper, the answer is clearly 'No.' It is not a way to falsify the linearity critique, but only to circumvent it: one assumes exponential population growth to be an *outside* process that nevertheless drives the long-run dynamics of her model. And what seems realistic in empirical research, e.g., Jones' (1995) model explains historical data without imposing any linearity: $\hat{A} = \frac{\lambda \hat{L}}{1-\phi}$, where $\hat{L}$ is exogenous, is irrelevant when considering a *complete* economic theory. To this extent, semi-endogenous growth models are also not immune to the linearity critique. How can one say, that she has $\dot{L} = nL$ in her model, but it is *not her* assumption? That there is a linearity that drives the dynamics of the model, but it is *outside* of the model? Linearity critique applies to all models which contain $\dot{A} = gA$; hence, it also applies to all models which contain $\dot{L} = nL$. And the argument that "it is a biological fact of nature, that people reproduce in proportion to their number" (Jones 2003) unfortunately does not stand the test of endogenization of fertility: an unmotivated knife-edge assumption that the intertemporal elasticity of substitution in consumption should be exactly one, is called for (see Jones 2003, or Connolly and Peretto 2003).[13]

If steady-state exponential growth is a knife-edge condition, then what alternatives do we have? Instead of looking for the least "painful" knife-edge assumption, we could try to explain long-run growth without relying on steady-state analysis at all. However, very little has been done in this area. We can only quote Kremer (1993) and Jones (2001), who argue that the world's economic history over the very long run (since one million BC in Kremer; since 25000 BC in Jones) can

[13] One could conjecture, of course, that people at large do not optimize over the number of their children, but follow simple rules of thumb. There are at least two counter-arguments to this point, however. First, once one endows the agents of her model with utility-maximizing behavior, leaving some of their "propensities" exogenous for reasons other than simplification seems highly questionable. And second, the enormous fertility drop in the course of the Demographic Transition certainly does not look like a purely exogenous change in the applied rule of thumb, or in the propensity to have children.

be explained like a transitional process, and not a long-run phenomenon in the traditional steady-state sense.

The result of this paper disqualifies the linearity critique as a method of rejecting growth models that contain linear differential equations. We have shown that all models that generate exponential/geometrical steady-state growth contain knife-edge assumptions, in some form of linearity, singularity or degree-one homogeneity.

On the other hand, the theorem emphasizes fragility of exponential/geometrical steady-state growth, arising due to its great sensitivity to disturbances in parameter values.

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